Summation problem

Find

$$\sum_{n=1}^{100} \frac{n}{n^4+4}$$

(1) Factorization

$$n^4+4 = (n^4+4n^2+4) - 4n^2 = (n^2+2)^2 - (2n)^2$$
$$= (n^2+2n+2)(n^2-2n+2)$$

(2) Partial Fraction

$$\frac{n}{n^4+4} = \frac{A}{n^2+2n+2} + \frac{\mathbf{B}}{n^2-2n+2}$$

$$n = A(n^2-2n+2) + B(n^2+2n+2)$$

$$A = -\frac{1}{4}, \quad B = \frac{1}{4}$$

Hence $\quad \frac{n}{n^4+4} = \frac{1}{4}\left(\frac{\mathbf{1}}{n^2-2n+2} - \frac{1}{n^2+2n+2}\right) = \frac{1}{4}\left[\frac{1}{(n-1)^2+1} - \frac{1}{(n+1)^2+1}\right]$

(3) Summation

$$\sum_{n=1}^{100} \frac{n}{n^4+4} = \frac{1}{4}\sum_{n=1}^{100}\left[\frac{1}{(n-1)^2+1} - \frac{1}{(n+1)^2+1}\right]$$

$$= \frac{1}{4}\left\{\left[\frac{1}{0^2+1} - \frac{1}{2^2+1}\right] + \left[\frac{1}{1^2+1} - \frac{1}{3^2+1}\right] + \left[\frac{1}{2^2+1} - \frac{1}{4^2+1}\right] + \cdots + \left[\frac{1}{97^2+1} - \frac{1}{99^2+1}\right] + \left[\frac{1}{98^2+1} - \frac{1}{100^2+1}\right] + \left[\frac{1}{99^2+1} - \frac{1}{101^2+1}\right]\right\}$$

$$= \frac{1}{4}\left\{\frac{1}{0^2+1} + \frac{1}{1^2+1} - \frac{1}{100^2+1} - \frac{1}{101^2+1}\right\}$$

$$= \frac{1}{4}\left\{1 + \frac{1}{2} - \frac{1}{10001} - \frac{1}{10202}\right\} \approx \underline{\underline{0.374950497501}}$$

Further interest point:

$$\sum_{n=1}^{\infty} \frac{n}{n^4+4} = \frac{3}{8} = 0.375$$

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